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1982 J. Phys. A: Math. Gen. 15 253

(http://iopscience.iop.org/0305-4470/15/1/034)

J. Phys. A: Math. Gen. 15 (1982) 253–268. Printed in Great Britain

First-order formulation of massive spin-2 field theories

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Received 26 May 1981

Abstract. First-order formulations of massive spin-2 theories are studied which involve only third and lower rank tensors. A systematic procedure is given for obtaining the equations in tensor form. If repeated irreps (RIR) are not allowed only essentially three distinct formulations are obtained. Some comments on their electromagnetic couplings are made. The use of the procedure when RIR are allowed is illustrated by deriving a one (complex) parameter family of Lagrangians which includes the conventional 'Palatini' first-order formulation of spin 2. The minimal polynomials of the L_0 matrices for the various theories are discussed, illustrating graphical methods for examining the nilpotency indices of a first-order theory.

1. Introduction

The theory of the massive spin-2 field has received much attention over the years since the initial construction of a Lagrangian formulation by Fierz and Pauli (1939). The original Fierz–Pauli Lagrangian for spin 2 was second order in derivatives and involved a second rank symmetric traceless tensor, and a scalar auxiliary field. Fierz and Pauli also considered the minimal coupling of the theory to an electromagnetic field. Federbush (1961) showed that to avoid a loss of constraints problem the minimal coupling had to be supplemented by a direct non-minimal coupling to the electromagnetic field strength. There followed a number of works on modification or generalisations of the Fierz–Pauli theory (Rivers 1964, Nath 1965, Bhargava and Watanabe 1966, Tait 1972, Reilly 1974). At the same time interest in general high-spin fields was generated by the discovery of the now well known inconsistency problems of Johnson and Sudarshan (1961) and Velo and Zwanzinger (1969a). In the course of investigating their acausality problem for spin other than $\frac{3}{2}$, Velo and Zwanzinger rediscovered the spin-2 loss of constraints problem, but were not at first aware of the non-minimal term solution of it. Velo (1972) later made a thorough analysis of the external field problem for the 'correct' non-minimally coupled spin-2 theory, showing that it too is acausal.

All of the work mentioned above dealt with a second-order formalism for the spin-2 theory. Much of the confusion which arose over this theory could be traced to the so-called 'derivative ordering ambiguity' (Nagpal 1973) which is a feature of second-order systems. This problem can be avoided by working from the start with a first-order formalism, the theory of which is very well developed (Gel'fand *et al* 1963), and for which the minimal coupling procedure is unambiguous. Early versions of such a theory were those of Adler (1966) and Deser *et al* (1966) who worked by analogy with the well known Palatini formalism for the gravitational field. This theory has 50 field components (of which only 5 must be independent for a spin-2 field!), and as we see in § 6, it is rather complicated. A more economical theory is the 30-component Schwinger–Chang theory (Schwinger 1963, Chang 1966) studied by Hagen (1972; see also Mathews *et al* 1980). This is the theory of § 3. As shown by Hagen (1972), the interaction behaviour of the theory is particularly sensitive to which first-order formalism one uses, and indeed one of the difficulties of the high-spin problem is that one can never (it seems) be completely sure that there is not some sufficiently complicated formulation of the theory which will not suffer from the normal ills of the external field problem. One of the early successes of supergravity was in the resolution of the acausality problem of Velo and Zwanzinger for the spin-$\frac{3}{2}$ field (Deser and Zumino 1976) and this gave hope that the high-spin pathologies could be circumvented. However, supergravity itself is now temporarily stuck at spin $\frac{5}{2}$ (Van Nieuwenhuizen 1981), essentially with a high-spin coupling problem. Further, while supergravity does avoid the spin-$\frac{3}{2}$ problem, it does not really attend to the question of why the high-spin pathologies should occur, or indeed if they *do* occur for all high-spin theories. Thus it appears that the high-spin problem is still open, although attention is focused at present on high-spin gauge theories.

In view of the sensitivity of the first-order formalism to the coupling scheme it is useful to know what are the possible first-order formalisms for a given spin and what is the nature of their constraint structure. Thus, Shamaly and Capri (1973) have investigated possible first-order spin-1 theories, and obtained schemes unifying the different formulations known previously. In this paper we perform the same service for spin 2, as a means of illustrating a general approach in which simple graphical arguments are used to make an initial sorting of the possibilities, followed up by a detailed constraint analysis in tensor form. If irreps of the proper Lorentz group ($\mathscr{L}_p$) are not allowed to be repeated, we find only three distinct first-order theories utilising tensors up to rank three. If repeated irreps (RIR) are allowed many further possibilities arise—of which the theory of Deser *et al*, considered in § 6, is one.

The theories we consider are of the general form

$$(L_\mu \partial^\mu + im)\psi = 0 \tag{1.1}$$

and are assumed to be $\mathscr{L}_p$ and space reflection covariant, and derivable from a real Lagrangian. The theory of such equations may be found in Gel'fand *et al* (1963), and the graphical techniques used in their analysis are described by Cox (1974a, b, c, 1978). For constraint and interaction analysis the form (1.1) is perhaps not so convenient as the usual tensor form, to which we resort to obtain the exact detailed form of the theories. However, recently much progress has been made in the external field problem for (1.1) (Mathews *et al* 1980), which involves knowledge of the minimal polynomial of L_0 in (1.1). To this end in § 7 we have used some simple graphical ideas (Cox 1981) to identify the possible minimal polynomials for the various theories.

We have only briefly discussed the external field problem for the theories, because for those of §§ 3 and 6 the results are already known (Hagen 1972) while the theories of §§ 4 and 5 are very complicated and are deferred to a separate publication.

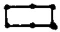
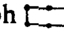
2. The method of identifying possible theories

We consider spin-2 theories described by a first-order equation of the form (1.1) which are covariant under space reflections and proper Lorentz transformations, and which can be derived from a Lagrangian. In particular we shall consider general theories which use at most third-rank tensors. The motivation for this is that, as is conventional, we shall eventually want our first-order equation to reduce to a second-order system in a second-rank tensor, and if we use at most third-rank tensors this can be done in at most one derivative step. Since the general third-rank tensor transforms according to

$$[\mathscr{D}(\tfrac{1}{2}\tfrac{1}{2})]^3 \sim \mathscr{D}(\tfrac{3}{2}\tfrac{3}{2}) \oplus 2\mathscr{D}(\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}) \oplus 4\mathscr{D}(\tfrac{1}{2}\tfrac{1}{2})$$

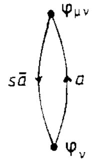
we are only interested in theories containing just the $\mathscr{L}_{\mathrm{p}}$ irreps appearing on the right-hand side and also the lower-rank irreps $\mathscr{D}(11)$, $\mathscr{D}(101)$, $\mathscr{D}(00)$.

To obtain maximum simplicity of the theories we shall initially restrict ourselves to those in which none of the above irreps of $\mathscr{L}_{\mathrm{p}}$ appears more than once—we discuss theories with repeated irreps (RIR) in § 6.

Using the graph-theoretical techniques of previous papers (Cox 1974a, b, c, 1978) the problem is to find which graphs, constructed on vertices corresponding to $\mathscr{L}_{\mathrm{p}}$ irreps, the edges corresponding to derivatives linking the irreps according to the non-zero elements of L_0, produce theories in which the spin-0 and 1 blocks are nilpotent and the spin-2 block has eigenvalues ± 1, each once only, and otherwise zeros. Space reflection covariance demands that these graphs (and the weights of their edges) be symmetric about the l_1 axis, so the total number of graphs is quite restricted. Further, some apparent possibilities can be immediately excluded by the topological nature of the graph. Thus for example any graph containing the subgraph as the 0 block cannot be allowed, because it has a unique maximal matching and therefore cannot be nilpotent (Cox 1974c). Such arguments exclude a number of graphs. Also, direct calculation on the general 1 block given in Cox (1974b) shows that neither of the 1 blocks  or  is allowed as they cannot be nilpotent. Shamaly and Capri (1971) have given a maximum spin-2 theory with precisely the latter graph , but their theory is in fact not manifestly space reflection covariant—the weights of the edges are not symmetric about the l_1 axis.

Using such ideas as above, it is found that only the graphs  remain as possibilities to be checked in detail. This could be done by using the known forms of the s blocks with arbitrary elements, constructing the L_μ and analysing the mass–spin spectra by standard methods for equation (1.1), choosing the elements of L_μ to give the required spectra. This is done by making the 0 and 1 blocks nilpotent and requiring the 2 block to have eigenvalues ± 1, each once and otherwise zero eigenvalues. This, for a given graph, will determine the L_0 and L_i matrices essentially uniquely. However, the study of the constraint structure and behaviour under interactions is perhaps more familiar if the equations are written in tensor form, rather than the form (1.1) (but see Mathews *et al* 1980). While the transition from (1.1) to tensor form is in principle straightforward (e.g. Frank 1973), it is much easier in practice to start directly from the graphs, with the $\mathscr{L}_{\mathrm{p}}$ irreps corresponding to the vertices represented by appropriate Lorentz tensors. Thus, once the graph-theoretical ideas have done the initial work of deciding which linkages are allowed between Lorentz irreps, then at that stage we resort to more usual tensor representations to obtain the detailed coefficients of the first-order equations.

The procedure is to represent each vertex in irreducible tensor form and link these tensors by derivatives in accordance with the structure of the graph. A directed edge from irrep i to irrep j denotes the construction of a term transforming according to irrep j from a term transforming according to irrep i and a single derivative (thus the edge represents a term in the Clebsch–Gordan expansion of $\mathscr{D}(\frac{1}{2}\,\frac{1}{2})\times$ irrep i), multiplied by an arbitrary complex number. For example, consider the edges linking irreps $\mathscr{D}(\frac{1}{2}\,\frac{1}{2})$, $\mathscr{D}(1\ 1)$:



Here $\mathscr{D}(1\ 1)$ is represented by a symmetric traceless tensor $\varphi_{\mu\nu}$, $\mathscr{D}(\frac{1}{2}\frac{1}{2})$ by a vector φ_μ. Then this graph represents the equations

$$\begin{aligned}\varphi_{\mu\nu} &= a\{\partial_\mu\varphi_\nu\}_{\mathrm{ST}}\\ &=\tfrac{1}{2}a[\partial_\mu\varphi_\nu+\partial_\nu\varphi_\mu-\tfrac{1}{2}g_{\mu\nu}(\partial\varphi)]\end{aligned}$$

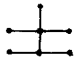
and

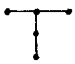
$$\varphi_\mu = s\bar{a}\,\partial^\nu\varphi_{\mu\nu}$$

where $\partial\varphi=\partial^\rho\varphi_\rho$ and $\{\ \}_{\mathrm{ST}}$ means take the symmetric traceless part. Also, $s=\pm1$ and the values a, $s\bar{a}$ are taken to ensure that the theory is derivable from a real Lagrangian—thus the above equations come from varying $\bar{\varphi}_{\mu\nu}$, $\bar{\varphi}_\mu$ in the Lagrangian term

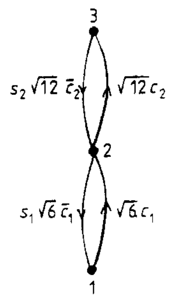
$$a\overline{\varphi}^{\mu\nu}\{\partial_\mu\varphi_\nu\}_{\mathrm{ST}}+s\bar{a}\overline{\varphi}^{\mu}(\partial^\nu\varphi_{\mu\nu})+\overline{\varphi}^{\mu\nu}\varphi_{\mu\nu}+\overline{\varphi}^{\mu}\varphi_\mu$$

which, it can be verified, is real up to a total divergence.

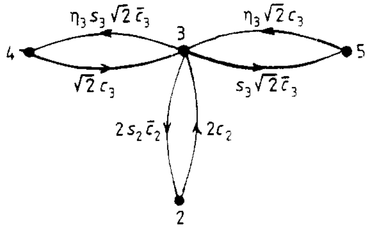
Once the general structure of the first-order system is written down, we then perform a constraint analysis to ensure that a unique spin-2 theory is obtained. This determines, in some cases uniquely, the values of the complex coefficients. That there must exist a solution giving such a theory in any particular case is most easily seen from the s-block analysis. In the next section we illustrate the above procedure in detail for the graph , and in §§ 4 and 5 we outline the corresponding results for the other types of graph, , 

3. The graph 

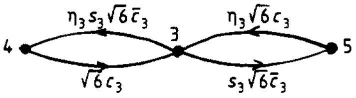
The s blocks for this graph are (Cox 1974b)



0 block



1 block



2 block

Here the notations are as in Cox (1974b). η_3 and the s_i are ± 1, and the c_i are arbitrary complex coefficients. In $\mathscr{D}(k, l)$ notation irreps 1, 2, 3 are $\mathscr{D}(0, 0)$, $\mathscr{D}(\frac{1}{2}\frac{1}{2})$, $\mathscr{D}(1, 1)$ respectively, while 4 and 5 are $\mathscr{D}(\frac{1}{2}, \frac{3}{2})$ and $\mathscr{D}(\frac{3}{2}, \frac{1}{2})$, which we collect together into the notation $\mathscr{D}(\frac{3}{2}\frac{1}{2}\frac{3}{2})$.

The unique spin-2 conditions are:

0-block nilpotency

$$6s_1|c_1|^2 + 12s_2|c_2|^2 = 0;$$

1-block nilpotency

$$4s_2|c_2|^2 + 4\eta_3 s_3|c_3|^2 = 0;$$

unique mass for spin-2 block

$$12\eta_3 s_3|c_3|^2 \neq 0.$$

It is easily verified that these equations have a unique solution for $s_1|c_1|^2$, $s_2|c_2|^2$ in terms of $\eta_3 s_3|c_3|^2$, and so a possible theory exists for this graph, and it is essentially unique—only the moduli of the coefficients c_i are determined, but their phases can always be assigned by some convenient convention.

To obtain the precise form of the theory, we realise the irreps involved by appropriate irreducible Lorentz tensors. Throughout this paper we realise the irreps $\mathscr{D}(0\,0)$, $\mathscr{D}(\frac{1}{2}\frac{1}{2})$, $\mathscr{D}(1, 1)$, $\mathscr{D}(\frac{3}{2}, \frac{3}{2})$ by tensors φ, φ_μ, $\varphi_{\mu\nu}$, $\varphi_{\mu\nu\rho}$ respectively, where $\varphi_{\mu\nu}$, $\varphi_{\mu\nu\rho}$ are both completely symmetric and traceless. We realise $\mathscr{D}(101) \sim \mathscr{D}(10) + \mathscr{D}(01)$ by an antisymmetric tensor $A_{\mu\nu}$. $\mathscr{D}(\frac{3}{2}\frac{1}{2}\frac{3}{2})$ is realised by a tensor $G_{\mu,\nu\rho}$ which is antisymmetric in ν and ρ and which satisfies the conditions

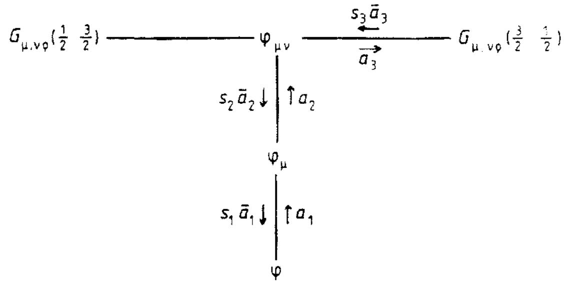
$$\varepsilon^{\mu\nu\rho\alpha} G_{\mu,\nu\rho} = 0 \tag{3.1}$$

$$g^{\mu\nu} G_{\mu,\nu\rho} = 0 \tag{3.2}$$

(these two conditions serve to eliminate the two vector parts in the transformation law $\mathscr{D}(\frac{1}{2}\frac{1}{2}) \otimes \mathscr{D}(101) \sim \mathscr{D}(\frac{3}{2}\frac{1}{2}\frac{3}{2}) \oplus 2\mathscr{D}(\frac{1}{2}\frac{1}{2})$ of a $T_{\mu,\nu\rho}$ antisymmetric in $\nu\rho$ (Singh and Hagen 1974)). Note that the symmetry and trace conditions on the fields are assumed *a priori*, for convenience. There is no loss of generality in this; being simply algebraic conditions, they can be easily incorporated into the Lagrangian without altering the theory, if required. Also, note that given an arbitrary third-rank tensor $V_{\mu\nu\rho}$, then a tensor $T_{\mu,\nu\rho}$ with the properties of $G_{\mu,\nu\rho}$ stated above can be constructed as follows:

$$\begin{aligned} T_{\mu,\nu\rho} = [V_{\mu\nu\rho}]_{\text{AST}} = {} & V_{\mu\nu\rho} + V_{\nu\mu\rho} - V_{\mu\rho\nu} - V_{\rho\mu\nu} - \tfrac{1}{3} g_{\mu\nu}[2V^\alpha{}_{\alpha\rho} - V^\alpha{}_{\rho\alpha} - V_\rho{}^\alpha{}_\alpha] \\ & + \tfrac{1}{3} g_{\mu\rho}(2V^\alpha{}_{\alpha\nu} - V^\alpha{}_{\nu\alpha} - V_\nu{}^\alpha{}_\alpha). \end{aligned} \tag{3.3}$$

It is easily verified that this object, antisymmetric in $\nu\rho$, satisfies the conditions (3.1), (3.2).



Having realised the irreps by appropriate tensors, we have to construct the general first-order system of equation with the linkage scheme

The equations are

$$G_{\mu,\nu\rho} = a_3\{\partial_\nu\varphi_{\mu\rho}\}_{\mathrm{AST}} \tag{3.4a}$$

$$\varphi_{\mu\nu} = s_3\bar{a}_3\{\partial^\rho G_{\mu,\nu\rho})_{\mathrm{ST}} + a_2\{\partial_\mu\varphi_\nu)_{\mathrm{ST}} \tag{3.4b}$$

$$\varphi_\mu = s_2\bar{a}_2\partial^\nu\varphi_{\mu\nu} + a_1\partial_\mu\varphi \tag{3.4c}$$

$$\varphi = s_1\bar{a}_1\partial^\mu\varphi_\mu \tag{3.4d}$$

where, from (3.3),

$$\{\partial_\nu\varphi_{\mu\rho}\}_{\mathrm{AST}} = \partial_\nu\varphi_{\mu\rho} - \partial_\rho\varphi_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}(\partial\varphi)_\rho + \tfrac{1}{2}g_{\rho\mu}(\partial\varphi)_\nu \tag{3.5}$$

where

$$(\partial\varphi)_\rho = \partial^\alpha\varphi_{\alpha\rho}.$$

We can reduce this to a pair of second-order equations in $\varphi_{\mu\nu}$ and φ_μ by eliminating the 'external' representations $G_{\mu,\nu\rho}$ and φ, and the result is

$$\varphi_{\mu\nu} = s_3|a_3|^2[\tfrac{2}{3}(\partial_\mu(\partial\varphi)_\nu + \partial_\nu(\partial\varphi)_\mu) - \partial^2\varphi_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}\partial^\rho(\partial\varphi)_\rho] + \tfrac{1}{2}a_2[\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu - \tfrac{1}{2}g_{\mu\nu}(\partial\varphi)] \tag{3.6a}$$

$$\varphi_\mu = s_2\bar{a}_2\partial^\nu\varphi_{\mu\nu} + s_1|a_1|^2\partial_\mu(\partial\varphi). \tag{3.6b}$$

This system will describe a propagating spin-2 field if it implies the constraints $\varphi_\mu = \partial\varphi = (\partial\varphi)_\mu = \partial^\rho(\partial\varphi)_\rho = 0$ and thereby reduces to the Klein–Gordon equation for $\varphi_{\mu\nu}$ (the original technique of Fierz and Pauli (1939)). Differentiating (3.6a) twice and (3.6b) once and putting $A = \partial\varphi$, $B = \partial^\rho(\partial\varphi)_\rho$ yields the equations

$$B = \tfrac{3}{4}a_2\partial^2 A$$

$$A = s_2\bar{a}_2B + s_1|a_1|^2\partial^2 A = (\tfrac{3}{4}s_2|a_2|^2 + s_1|a_1|^2])\partial^2 A,$$

whence $A = 0$ results if we choose

$$\tfrac{3}{4}s_2|a_2|^2 + s_1|a_1|^2 = 0. \tag{3.7}$$

This ensures

$$\partial\varphi = \partial^\rho(\partial\varphi)_\rho = 0. \tag{3.8}$$

Differentiating (3.6a) once, using (3.8), gives

$$(\partial\varphi)_\mu = -\tfrac{1}{3}s_3|a_3|^2\partial^2(\partial\varphi)_\mu + \tfrac{1}{2}a_2\partial^2\varphi_\mu,$$

which, with (3.6*b*), gives

$$(\partial\varphi)_\mu = (\tfrac{1}{2}s_2|a_2|^2 - \tfrac{1}{3}s_3|a_3|^2)\partial^2(\partial\varphi)_\mu = 0$$

and to obtain $(\partial\varphi)_\mu = 0$ we must choose

$$\tfrac{1}{2}s_2|a_2|^2 - \tfrac{1}{3}s_3|a_3|^2 = 0, \tag{3.9}$$

which ensures, using (3.6*b*), that

$$(\partial\varphi)_\mu = \varphi_\mu = 0. \tag{3.10}$$

With (3.7)–(3.10), (3.6*a*) reduces to

$$\left(\partial^2 + \frac{1}{s_3|a_3|^2}\right)\varphi_{\mu\nu} = 0. \tag{3.11}$$

For real mass we must therefore take $s_3 = -1$, and for simplicity we shall assume unit mass and therefore take $|a_3|^2 = 1$. Then from (3.7), (3.9) we have

$$s_2 = -1 \qquad s_1 = +1 \qquad |a_2|^2 = \tfrac{2}{3} \qquad |a_1|^2 = \tfrac{1}{2}. \tag{3.12}$$

Without loss of generality, we will take the a_i to be real and positive; then the desired equations (3.5*a*)–(3.5*d*) become

$$G_{\mu,\nu\rho} = \partial_\nu\varphi_{\mu\rho} - \partial_\rho\varphi_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}(\partial\varphi)_\rho + \tfrac{1}{3}g_{\rho\mu}(\partial\varphi)_\nu \tag{3.13a}$$

$$\varphi_{\mu\nu} = -\tfrac{1}{2}(\partial^\rho G_{\mu,\nu\rho} + \partial^\rho G_{\nu,\mu\rho}) + \tfrac{1}{6}\sqrt{6}[\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu - \tfrac{1}{2}g_{\mu\nu}(\partial\varphi)] \tag{3.13b}$$

$$\varphi_\mu = -\sqrt{\tfrac{2}{3}}\,\partial^\nu\varphi_{\mu\nu} + \tfrac{1}{2}\sqrt{2}\,\partial_\mu\varphi \tag{3.13c}$$

$$\varphi = \tfrac{1}{2}\sqrt{2}\,\partial^\mu\varphi_\mu. \tag{3.13d}$$

This theory is in fact already well known in a slightly different form (Schwinger 1963, Chang 1967, Hagen 1972). The Schwinger–Chang theory is a first-order system involving a second-rank symmetric tensor $h_{\mu\nu}$ not traceless, and a third-rank tensor $H_{\mu,\nu\rho} = -H_{\mu,\rho\nu}$ with just the property (3.1):

$$\varepsilon^{\mu\nu\rho\alpha}H_{\mu,\nu\rho} = 0. \tag{3.14}$$

$H_{\mu,\nu\rho}$ transforms according to the representation $\mathscr{D}(\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2})\oplus\mathscr{D}(\tfrac{1}{2}\tfrac{1}{2})$. To obtain the Schwinger–Chang theory from (3.13) we absorb $\varphi_{\mu\nu}$ and φ into $h_{\mu\nu}$ and $G_{\mu,\nu\rho}$, φ_μ into $H_{\mu,\nu\rho}$. Specifically, with

$$\sqrt{6}\varphi_\mu = H^\alpha{}_{,\alpha\mu} \tag{3.15a}$$

$$\varphi_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}h \tag{3.15b}$$

$$G_{\mu,\nu\rho} = H_{\mu,\nu\rho} - \tfrac{1}{3}g_{\mu\nu}H^\alpha{}_{,\alpha\rho} + \tfrac{1}{3}g_{\mu\rho}H^\alpha{}_{,\alpha\nu} \tag{3.15c}$$

$$h = h^\mu{}_\mu = \sqrt{\tfrac{2}{3}}\partial^\rho\varphi_\rho = \tfrac{2}{3}\sqrt{3}\,\varphi \tag{3.15d}$$

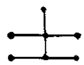
the equations (3.13) take the form

$$H_{\mu,\nu\rho} = \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu} + g_{\mu\nu}(\partial_\rho h - (\partial h)_\rho) - g_{\mu\rho}(\partial_\nu h - (\partial h)_\nu) \tag{3.16a}$$

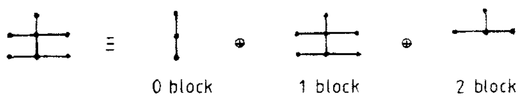
$$h_{\mu\nu} - g_{\mu\nu}h = -\tfrac{1}{2}\partial^\rho(H_{\mu,\nu\rho} + H_{\nu,\mu\rho}) \tag{3.16b}$$

which are the Schwinger–Chang equations for unit mass (Hagen 1972). This observation renders the analysis of the minimally coupled form of (3.13) unnecessary since in the form (3.16) it has been done by Hagen (1972) and Kobayashi and Shamaly (1978).

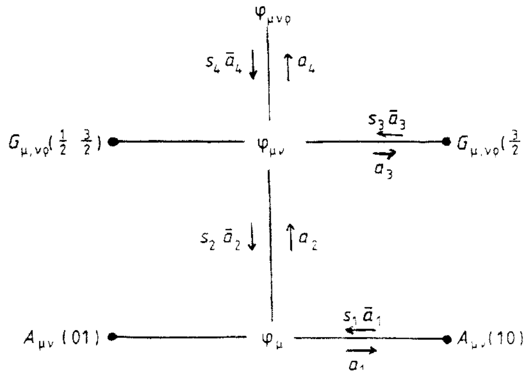
However, it is interesting to note that the interaction analysis of (3.16) is much easier than that of the form (3.13), the reason being that (3.16) reduces directly to a second-order equation in the single field $h_{\mu\nu}$ on eliminating $H_{\mu,\nu\rho}$, whereas retaining the form (3.13) makes this less transparent—the number of possible interaction terms is much increased and more difficult to handle. Thus, an efficiency in the number and form of tensors used in the theory is advantageous, irrespective of what Lorentz irreps are involved. In respect of (3.16), the conclusion of Hagen (1972) was that it suffers no loss of constraints problem on minimal coupling, while Kobayashi and Shamaly (1978) showed that even so, the theory was acausal—which in fact had already been shown by Velo (1972).

4. The graph 

For this theory the s-block decomposition is



and writing down the nilpotency conditions for the 0 block and 1 block and the unique mass condition for the 2 block it can be verified that, just as in the previous section, there is an essentially unique solution for the coefficients of this theory. This encourages us to try the linkage scheme



It yields the equations

$$\varphi_{\mu\nu\rho} = a_4\{\partial_\mu\varphi_{\nu\rho}\}_{\mathrm{ST}} \tag{4.1a}$$

$$G_{\mu,\nu\rho} = a_3\{\partial_\nu\varphi_{\mu\rho}\}_{\mathrm{AST}} \tag{4.1b}$$

$$\varphi_{\mu\nu} = s_4\bar{a}_4\{\partial^\rho\varphi_{\mu\nu\rho}\}_{\mathrm{ST}} + s_3\bar{a}_3\{\partial^\rho G_{\mu,\nu\rho}\}_{\mathrm{ST}} + a_2\{\partial_\mu\varphi_\nu\}_{\mathrm{ST}} \tag{4.1c}$$

$$\varphi_\mu = s_2\bar{a}_2\partial^\rho\varphi_{\mu\rho} + s_1\bar{a}_1\partial^\rho A_{\mu\rho} \tag{4.1d}$$

$$A_{\mu\nu} = a_1(\partial_\mu\varphi_\nu - \partial_\nu\varphi_\mu). \tag{4.1e}$$

In detail, the first three equations are

$$\varphi_{\mu\nu\rho} = a_4[\tfrac{1}{3}(\partial_\mu\varphi_{\nu\rho} + \partial_\nu\varphi_{\mu\rho} + \partial_\rho\varphi_{\mu\nu}) - \tfrac{1}{9}(g_{\mu\nu}(\partial\varphi)_\rho + g_{\mu\rho}(\partial\varphi)_\nu + g_{\nu\rho}(\partial\varphi)_\mu)] \tag{4.2a}$$

$$G_{\mu,\nu\rho} = a_3[\partial_\nu\varphi_{\mu\rho} - \partial_\rho\varphi_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}(\partial\varphi)_\rho + \tfrac{1}{3}g_{\rho\mu}(\partial\varphi)_\nu] \tag{4.2b}$$

$$\varphi_{\mu\nu} = s_4\bar{a}_4\partial^\rho\varphi_{\mu\nu\rho} + \tfrac{1}{2}s_3\bar{a}_3(\partial^\rho G_{\mu,\nu\rho} + \partial^\rho G_{\nu,\mu\rho}) + \tfrac{1}{2}a_2(\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu - \tfrac{1}{2}g_{\mu\nu}(\partial\varphi)). \tag{4.2c}$$

Substituting for $\varphi_{\mu\nu\rho}$ and $G_{\mu,\nu\rho}$ in $\varphi_{\mu\nu}$ gives the equation

$$\begin{aligned}\varphi_{\mu\nu} = {}&(\tfrac{2}{9}x_4 + \tfrac{2}{3}x_3)(\partial_\mu(\partial\varphi)_\nu + \partial_\nu(\partial\varphi)_\mu) + (\tfrac{1}{3}x_4 - x_3)\partial^2\varphi_{\mu\nu} \\ &-(\tfrac{1}{9}x_4 + \tfrac{1}{3}x_3)g_{\mu\nu}\partial^\rho(\partial\varphi)_\rho + \tfrac{1}{2}a_2(\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu - \tfrac{1}{2}g_{\mu\nu}(\partial\varphi))\end{aligned} \tag{4.3a}$$

where $x_i = s_i|a_i|^2$.

On the other hand, substituting for $A_{\mu\nu}$ in the equation for φ_μ we obtain

$$\varphi_\mu = s_2\bar{a}_2(\partial\varphi)_\mu + x_1(\partial_\mu(\partial\varphi) - \partial^2\varphi_\mu). \tag{4.3b}$$

We now have to choose the a_i, s_i such that (4.3) reduce to the constraints $(\partial\varphi) = \partial^\rho(\partial\varphi)_\rho = \varphi_\mu = (\partial\varphi)_\mu = 0$ and yield a Klein–Gordon equation for $\varphi_{\mu\nu}$.

From (4.3a), with $A = \partial\varphi$, $B = \partial^\rho(\partial\varphi)_\rho$, we obtain by differentiating twice

$$B = \tfrac{2}{3}x_4\partial^2 B + \tfrac{3}{4}a_4\partial^2 A \tag{4.4a}$$

and by differentiating (4.3b) once

$$A = s_2\bar{a}_2 B. \tag{4.4b}$$

Equations (4.4) imply

$$B = (\tfrac{2}{3}x_4 + \tfrac{3}{4}x_2)\partial^2 B$$

so we must take

$$x_4 + \tfrac{9}{8}x_2 = 0 \tag{4.6}$$

and this will ensure that

$$\partial\varphi = \partial^\rho(\partial\varphi)_\rho = 0. \tag{4.7}$$

Now, by differentiating (4.3a) once, using (4.7), we obtain

$$(\partial\varphi)_\mu = (\tfrac{5}{9}x_4 - \tfrac{1}{3}x_3)\partial^2(\partial\varphi)_\mu + \tfrac{1}{2}a_2\partial^2\varphi_\mu \tag{4.8}$$

and the equation for φ_μ is

$$\varphi_\mu = s_2\bar{a}_2(\partial\varphi)_\mu - x_1\partial^2\varphi_\mu. \tag{4.9}$$

The equations (4.8), (4.9) yield $\varphi_\mu = (\partial\varphi)_\mu = 0$ only if the operator determinant is always non-zero,

$$\begin{vmatrix} (\tfrac{5}{9}x_4 - \tfrac{1}{3}x_3)\partial^2 - 1 & \tfrac{1}{2}a_2\partial^2 \\ s_2\bar{a}_2 & -(x_1\partial^2 + 1) \end{vmatrix} \neq 0.$$

This is so only if

$$\tfrac{5}{9}x_4 - \tfrac{1}{3}x_3 = 0 \tag{4.10a}$$

$$\tfrac{1}{2}x_2 - x_1 = 0 \tag{4.10b}$$

(these conditions make the ∂^4 and ∂^2 terms vanish). Equations (4.6), (4.10) have the unique solution

$$x_1 = \tfrac{1}{2}x_2 \qquad x_3 = -\tfrac{15}{8}x_2 \qquad x_4 = -\tfrac{9}{8}x_2 \tag{4.11}$$

and then (4.3a) reduces to the equation

$$\varphi_{\mu\nu} = \tfrac{3}{2}x_2\partial^2\varphi_{\mu\nu} \tag{4.12}$$

for $\varphi_{\mu\nu}$. So, for unit mass we can take $x_2 = \frac{2}{3}$. Then (4.1) gives, assuming the a_i to be real and positive,

$$s_1 = s_2 = -s_3 = -s_4 = +1$$
$$a_1 = \tfrac{1}{3}\sqrt{3} \qquad a_2 = \sqrt{\tfrac{2}{3}} \qquad a_3 = \tfrac{1}{2}\sqrt{5} \qquad a_4 = \tfrac{1}{2}\sqrt{3}. \tag{4.13}$$

With these values in (4.1) the resulting system describes a propagating spin-2 field with unit mass. It is convenient to rescale the fields as

$$\varphi_{\mu\nu\rho} \to \tfrac{2}{3}\sqrt{3}\,\varphi_{\mu\nu\rho} \qquad G_{\mu,\nu\rho} \to \tfrac{2}{5}\sqrt{5}\,G_{\mu,\nu\rho}$$
$$\varphi_\mu \to \sqrt{\tfrac{3}{2}}\,\varphi_\mu \qquad A_{\mu\nu} \to \tfrac{3}{2}\sqrt{2}\,A_{\mu\nu} \qquad \varphi_{\mu\nu}\ \text{unchanged}. \tag{4.14}$$

Then the equations (4.1), with (4.13), become

$$\varphi_{\mu\nu\rho} = \tfrac{1}{3}(\partial_\mu\varphi_{\nu\rho} + \partial_\nu\varphi_{\mu\rho} + \partial_\rho\varphi_{\mu\nu}) - \tfrac{1}{9}(g_{\mu\nu}(\partial\varphi)_\rho + g_{\mu\rho}(\partial\varphi)_\nu + g_{\nu\rho}(\partial\varphi)_\mu) \tag{4.15a}$$
$$G_{\mu,\nu\rho} = \partial_\nu\varphi_{\mu\rho} - \partial_\rho\varphi_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}(\partial\varphi)_\rho + \tfrac{1}{3}g_{\rho\mu}(\partial\varphi)_\nu \tag{4.15b}$$
$$\varphi_{\mu\nu} = -\tfrac{3}{4}\partial^\rho\varphi_{\mu\nu\rho} - \tfrac{5}{8}(\partial^\rho G_{\mu,\nu\rho} + \partial^\rho G_{\nu,\mu\rho}) + \tfrac{1}{3}(\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu - \tfrac{1}{2}g_{\mu\nu}(\partial\varphi)) \tag{4.15c}$$
$$\varphi_\mu = (\partial\varphi)_\mu + \tfrac{1}{3}\partial^\rho A_{\mu\rho} \tag{4.15d}$$
$$A_{\mu\nu} = \partial_\mu\varphi_\nu - \partial_\nu\varphi_\mu. \tag{4.15e}$$

From (4.15) follow the constraints $\partial\varphi = \partial^\rho(\partial\varphi)_\rho = \varphi_\mu = (\partial\varphi)_\mu = 0$ and then the equation for $\varphi_{\mu\nu}$ reduces to $\varphi_{\mu\nu} = \partial^2\varphi_{\mu\nu}$.

Now in the minimally coupled version of (4.15) the constraint analysis is complicated by the presence of the lower spin field φ_μ, and it seems sensible to adopt a similar approach to that in the previous section and absorb φ_μ and $G_{\mu,\nu\rho}$ into a single tensor $H_{\mu,\nu\rho}$. Also, at the same time we can incorporate $\varphi_{\mu\nu}$, $A_{\mu\nu}$ into a single general second-rank tensor $Z_{\mu\nu}$. Specifically we substitute

$$G_{\mu,\nu\rho} = H_{\mu,\nu\rho} - \tfrac{1}{3}g_{\mu\nu}H^\alpha{}_{,\alpha\rho} + \tfrac{1}{3}g_{\mu\rho}H^\alpha{}_{,\alpha\nu} \tag{4.16a}$$
$$\varphi_\mu = H^\alpha{}_{,\alpha\mu} \tag{4.16b}$$
$$\varphi_{\mu\nu} = \tfrac{1}{2}(Z_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}Z) \tag{4.16c}$$
$$B_{\mu\nu} = \tfrac{1}{2}(Z_{\mu\nu} + Z_{\nu\mu} - Z_{\nu\mu}) \tag{4.16d}$$

where $Z = Z^\mu{}_\mu$. Then the equations (4.15) become

$$\begin{aligned}\varphi_{\mu\nu\rho} = {} & \tfrac{1}{6}(\partial_\mu Z_{\nu\rho} + \partial_\mu Z_{\rho\nu} + \partial_\nu Z_{\mu\rho} + \partial_\nu Z_{\rho\mu} + \partial_\rho Z_{\mu\nu} + \partial_\rho Z_{\nu\mu}) \\ & - \tfrac{1}{18}g_{\nu\rho}\partial_\mu Z - \tfrac{1}{18}g_{\mu\rho}\partial_\nu Z - \tfrac{1}{18}g_{\mu\nu}\partial_\rho Z - \tfrac{1}{18}g_{\mu\nu}(\partial^\alpha Z_{\alpha\rho} + \partial^\alpha Z_{\rho\alpha}) \\ & - \tfrac{1}{18}g_{\mu\rho}(\partial^\alpha Z_{\alpha\nu} + \partial^\alpha Z_{\nu\alpha}) - \tfrac{1}{18}g_{\nu\rho}(\partial^\alpha Z_{\alpha\mu} + \partial^\alpha Z_{\mu\alpha})\end{aligned} \tag{4.17a}$$
$$\begin{aligned}H_{\mu,\nu\rho} = {} & \tfrac{1}{2}\partial_\nu(Z_{\mu\rho} + Z_{\rho\mu}) - \tfrac{1}{2}\partial_\rho(Z_{\mu\nu} + Z_{\nu\mu}) - \tfrac{1}{18}g_{\mu\nu}\partial^\alpha Z_{\rho\alpha} + \tfrac{1}{18}g_{\mu\nu}\partial^\alpha Z_{\alpha\rho} \\ & + \tfrac{1}{18}g_{\mu\rho}\partial^\alpha Z_{\nu\alpha} - \tfrac{1}{18}g_{\mu\rho}\partial^\alpha Z_{\alpha\nu} + \tfrac{1}{4}g_{\mu\nu}\partial_\rho Z - \tfrac{1}{4}g_{\mu\rho}\partial_\nu Z\end{aligned} \tag{4.17b}$$
$$Z_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}Z = -\tfrac{3}{4}\partial^\rho\varphi_{\mu\nu\rho} - \tfrac{5}{8}(\partial^\rho H_{\mu,\nu\rho} + \partial^\rho H_{\nu,\mu\rho}) - \tfrac{7}{8}\partial_\mu H^\alpha{}_{,\alpha\nu} + \tfrac{9}{8}\partial_\nu H^\alpha{}_{,\alpha\mu} + \tfrac{1}{4}g_{\mu\nu}\partial^\rho H^\alpha{}_{,\alpha\rho}. \tag{4.17c}$$

It can be verified that this system leads to the constraints

$$\partial^\mu \partial^\nu (Z_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} Z) = 0$$

$$\tfrac{1}{2}\partial^\rho (Z_{\mu\rho} + Z_{\rho\mu}) - \tfrac{1}{4}\partial_\mu Z = 0$$

$$\tfrac{1}{2}\partial^\rho (Z_{\mu\rho} - Z_{\rho\mu}) = 0$$

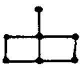
resulting in an equation for

$$\varphi_{\mu\nu} = \tfrac{1}{2}(Z_{\mu\nu} + Z_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} Z)$$

of the form $\varphi_{\mu\nu} = \partial^2 \varphi_{\mu\nu}$, as required.

The system (4.15), or (4.17), seems to be a new formulation of the massive spin-2 field in terms of a non-symmetric tensor $Z_{\mu\nu}$. Recently a new massless spin-2 theory has been given (Moffat 1979) which utilises a non-symmetric second-rank tensor, and it would be interesting to compare the zero-mass limit of (4.17), or indeed the theory of the next section, with this new theory.

The minimally coupled version of (4.17), $\partial_\mu \to \partial_\mu - \mathrm{i}eA_\mu = \pi_\mu$, is easily and unambiguously written down, being a first-order system. Eliminating $\varphi_{\mu\nu\rho}$, $H_{\mu,\nu\rho}$ this leads to a second-order equation for $Z_{\mu\nu}$ which contains a large number of troublesome coupling terms with the electromagnetic field $F_{\mu\nu}$, some of which are derivative couplings. High-spin field theories can suffer a change in the number of constraints due to some forms of coupling to external fields (Fierz and Pauli 1939, Komar 1961, Federbush 1961, Velo and Zwanzinger 1969b, Velo 1972, Hagen 1972, Tait 1972, Jenkins 1974, Mathews *et al* 1976, 1980), and this will change the number of degrees of freedom of the field. In many cases, including some types of spin-2 theories, minimal electromagnetic coupling suffers from precisely this problem; for example, the theory of § 6 (Hagen 1972) does. For such theories one must introduce non-minimal coupling terms in the Lagrangian to cancel the unwanted terms. For the theory of (4.17) this will involve additional derivative couplings in the original Lagrangian, with all of the attendant problems of such couplings. Even then, the resulting interaction theory is almost certainly acausal, although only a detailed analysis of the interaction theory will confirm this. Nevertheless, (4.17) is a valid and unusual formulation of free-field spin-2 theory.

5. The graph 

An s-block analysis confirms that for this graph there are solutions to the nilpotency and mass conditions required for a massive spin-2 theory. The tensor equations are the same as (4.1), but with now a linkage between $A_{\mu\nu}$ and $G_{\mu,\nu\rho}$, so the equations are

$$\varphi_{\mu\nu\rho} = a_4\{\partial_\mu \varphi_{\nu\rho}\}_{\mathrm{ST}} \tag{5.1a}$$

$$G_{\mu,\nu\rho} = a_3\{\partial_\nu \varphi_{\mu\rho}\}_{\mathrm{AST}} + a_5\{\partial_\mu A_{\nu\rho})_{\mathrm{AST}} \tag{5.1b}$$

$$\varphi_{\mu\nu} = s_4 \bar{a}_4 \{\partial^\rho \varphi_{\mu\nu\rho}\}_{\mathrm{ST}} + s_3 \bar{a}_3 \{\partial^\rho G_{\mu,\nu\rho}\}_{\mathrm{ST}} + a_2\{\partial_\mu \varphi_\nu\}_{\mathrm{ST}} \tag{5.1c}$$

$$A_{\mu\nu} = a_1(\partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu) + s_5 \bar{a}_5 \partial^\rho G_{\rho,\mu\nu} \tag{5.1d}$$

$$\varphi_\mu = s_2 \bar{a}_2 \partial^\nu \varphi_{\mu\nu} + s_1 \bar{a}_1 \partial^\nu A_{\mu\nu}. \tag{5.1e}$$

By following through the same procedure as in §§ 3 and 4 we obtain a unique spin-2 unit mass theory if we satisfy the equations

$$x_1-\tfrac{1}{2}x_2+\tfrac{1}{3}x_3-\tfrac{5}{9}x_4=0 \qquad \tfrac{3}{4}x_2+\tfrac{2}{3}x_4=0$$
$$\tfrac{10}{9}x_4x_5+x_2x_5-\tfrac{5}{9}x_1x_4+\tfrac{1}{3}x_1x_3+C=0 \qquad \tfrac{1}{3}x_4-x_3=1$$

where

$$C=\tfrac{1}{3}s_1s_5a_2a_3\bar{a}_1\bar{a}_5+s_2s_3a_1a_5\bar{a}_2\bar{a}_3.$$

This system of equations is equivalent to

$$x_1=\tfrac{1}{3} \qquad x_2=-\tfrac{8}{9}x_4 \qquad x_3=\tfrac{1}{3}x_4-1$$

and

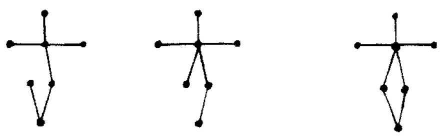
$$(\tfrac{2}{9}x_5-\tfrac{4}{27})x_4-\tfrac{1}{9}+C=0.$$

The last equation shows that here we have a one (complex) parameter family of solutions for the a_i and so there is a one-parameter family of systems of equations of the form (5.1) yielding a unique spin-2, unit mass, propagating field. While the interaction analysis of such a theory will be very difficult the extra freedom provided by the free parameter may allow some useful simplification.

6. Repeated irreducible representations

One of the frustrating aspects of high-spin theories is that by allowing the Lorentz irreps to be repeated, one can it seems obtain theories of almost unlimited complexity. Here we shall just illustrate the general approach to be followed in any particular case by means of an example which yields a theory already known. Govindarajan *et al* (1980) have recently begun to investigate the use of RIR in some particular general cases and their results may be of use here.

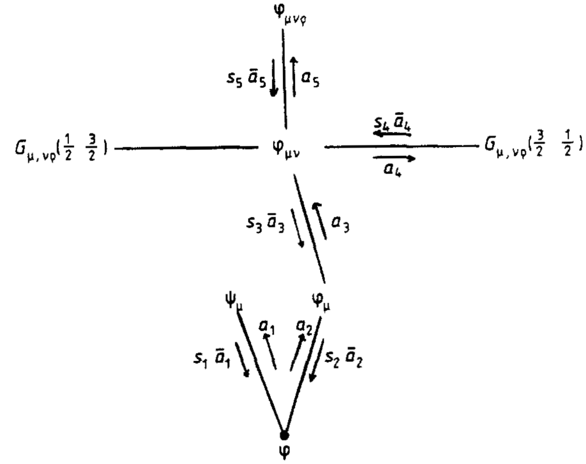
The theory we aim to reproduce is the 50-component theory of Deser *et al* (1966) described by Hagen (1972). This utilises a third-rank tensor $\Gamma_{\mu\nu,\rho}=\Gamma_{\nu\mu,\rho}$, transforming according to $\mathscr{D}(\frac{3}{2}\frac{3}{2})+\mathscr{D}(\frac{3}{2}\frac{1}{2}\frac{3}{2})+2\mathscr{D}(\frac{1}{2}\frac{1}{2})$ and a symmetric tensor $h_{\mu\nu}\sim\mathscr{D}(11)+\mathscr{D}(00)$. We therefore begin with these representations and consider possible graphs constructed with them. A little consideration reveals just three distinct possibilities:



(i) (ii) (iii)

It is (i) which corresponds to the theory of Deser *et al.* The reason we know this is because on contraction of the field equations of Deser *et al* it is seen that the two vector parts $\Gamma^{\mu}{}_{\mu,\nu}$, $\Gamma^{\mu}{}_{\nu,\mu}$ are both linked to the scalar part $h^{\mu}{}_{\mu}$ by differentiations, and further that $\Gamma^{\mu}{}_{\mu,\nu}$ alone is linked to the $\mathscr{D}(11)$ part of $h_{\mu\nu}$ by a derivative. The graphs (ii) and (iii) will yield alternative spin-2 formulations. It can be checked by s-block analysis that (i) does indeed allow a unique spin-2 theory.

The linkage scheme for (i) is



Using earlier results the corresponding equations are

$$\varphi_{\mu\nu\rho} = a_5[\tfrac{1}{3}(\partial_\mu\varphi_{\nu\rho} + \partial_\nu\varphi_{\mu\rho} + \partial_\rho\varphi_{\mu\nu}) - \tfrac{1}{9}(g_{\mu\nu}(\partial\varphi)_\rho + g_{\mu\rho}(\partial\varphi)_\nu + g_\rho(\partial\varphi)_\mu)] \tag{6.1a}$$

$$G_{\mu,\nu\rho} = a_4(\partial_\nu\varphi_{\mu\rho} - \partial_\rho\varphi_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}(\partial\varphi)_\rho + \tfrac{1}{3}g_{\rho\mu}(\partial\varphi)_\nu) \tag{6.1b}$$

$$\varphi_{\mu\nu} = s_5 a_5 \partial^\rho\varphi_{\mu\nu\rho} + \tfrac{1}{2}s_4\bar{a}_4(\partial^\rho G_{\mu,\nu\rho} + \partial^\rho G_{\nu,\mu\rho}) + \tfrac{1}{2}a_3(\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu - \tfrac{1}{2}g_{\mu\nu}(\partial\varphi)) \tag{6.1c}$$

$$\varphi_\mu = a_2\partial_\mu\varphi + s_3\bar{a}_3\partial^\rho\varphi_{\mu\rho} \tag{6.1d}$$

$$\psi_\mu = a_1\partial_\mu\varphi \tag{6.1e}$$

$$\varphi = s_1\bar{a}_1\partial^\mu\psi_\mu + s_2\bar{a}_2\partial^\mu\varphi_\mu. \tag{6.1f}$$

Eliminating $\varphi_{\mu\nu\rho}$, $G_{\mu,\nu\rho}$, φ gives the equations

$$\varphi_{\mu\nu} = (\tfrac{2}{9}x_5 + \tfrac{2}{3}x_4)(\partial_\mu(\partial\varphi)_\nu + \partial_\nu(\partial\varphi)_\mu) + (\tfrac{1}{3}x_5 - x_4)\partial^2\varphi_{\mu\nu}$$
$$-(\tfrac{1}{9}x_5 + \tfrac{1}{3}x_4)g_{\mu\nu}\partial^\rho(\partial\varphi)_\rho + \tfrac{1}{2}a_3(\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu - \tfrac{1}{2}g_{\mu\nu}(\partial\varphi)) \tag{6.2a}$$

$$\varphi_\mu = x_2\partial_\mu(\partial\varphi) + s_1\bar{a}_1 a_2\partial_\mu(\partial\psi) + s_3\bar{a}_3(\partial\varphi)_\mu \tag{6.2b}$$

$$\psi_\mu = x_1\partial_\mu(\partial\psi) + s_2 a_1\bar{a}_2\partial_\mu(\partial\varphi). \tag{6.2c}$$

In these three equations we saturate all indices with derivatives and obtain three equations for $\partial^\rho(\partial\varphi)_\rho$, $\partial\varphi$, $\partial\psi$. The conditions that these quantities vanish identically are (with $x_i = s_i|a_i|^2$)

$$x_1 + x_2 + \tfrac{3}{4}x_3 + \tfrac{2}{3}x_5 = 0 \tag{6.3a}$$

$$\tfrac{2}{3}x_1x_5 + \tfrac{2}{3}x_2x_5 + \tfrac{3}{4}x_1x_3 = 0. \tag{6.3b}$$

Then the corresponding equations for $(\partial\varphi)_\mu$, φ_μ, ψ_μ yield zero for these quantities only if

$$\tfrac{5}{9}x_5 - \tfrac{1}{3}x_4 + \tfrac{1}{2}x_3 = 0. \tag{6.3c}$$

With these conditions satisfied equation (6.2a) reduces to

$$\varphi_{\mu\nu} = (\tfrac{1}{3}x_5 - x_4)\partial^2\varphi_{\mu\nu}$$

and so for unit mass

$$\tfrac{1}{3}x_5 - x_5 = 1. \tag{6.3d}$$

With, say, $x_4 = \alpha$ the equations (6.3) yield the solution

$$x_1 = \frac{2(\alpha+1)}{4\alpha+5} \qquad x_2 = \frac{1}{2(4\alpha+5)}$$
$$x_3 = -\tfrac{2}{3}(4\alpha+5) \qquad x_4 = \alpha \qquad x_5 = 3(\alpha+1). \tag{6.4}$$

Rescaling the fields according to

$$\varphi_{\mu\nu\rho} \to \frac{1}{a_5}\varphi_{\mu\nu\rho} \qquad G_{\mu,\nu\rho} \to \frac{1}{a_4}G_{\mu,\nu\rho}$$
$$\varphi_\mu \to \frac{1}{s_3\bar{a}_3}\varphi_\mu \qquad \psi_\mu \to \frac{a_2}{a_1 s_3 \bar{a}_3}\psi_\mu \qquad \varphi \to \frac{a_2}{s_3\bar{a}_3}\varphi \tag{6.5}$$

the system (6.1) becomes, with (6.4),

$$\varphi_{\mu\nu\rho} = \tfrac{1}{3}(\partial_\mu\varphi_{\nu\rho} + \partial_\nu\varphi_{\mu\rho} + \partial_\rho\varphi_{\mu\nu}) - \tfrac{1}{9}(g_{\mu\nu}(\partial\varphi)_\rho + g_{\mu\rho}(\partial\varphi)_\nu + g_{\nu\rho}(\partial\varphi)_\mu) \tag{6.6a}$$
$$G_{\mu,\nu\rho} = \partial_\nu\varphi_{\mu\rho} - \partial_\rho\varphi_{\mu\nu} - \tfrac{1}{3}g_{\mu\nu}(\partial\varphi)_\rho + \tfrac{1}{3}g_{\rho\mu}(\partial\varphi)_\nu \tag{6.6b}$$
$$\varphi_{\mu\nu} = 3(\alpha+1)\partial^\rho\varphi_{\mu\nu\rho} + \tfrac{1}{2}\alpha(\partial^\rho G_{\mu,\nu\rho} + \partial^\rho G_{\nu,\mu\rho}) - \tfrac{1}{3}(4\alpha+5)(\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu - \tfrac{1}{2}g_{\mu\nu}(\partial\varphi)) \tag{6.6c}$$
$$\varphi_\mu = \partial_\mu\varphi + (\partial\varphi)_\mu \tag{6.6d}$$
$$\psi_\mu = \partial_\mu\varphi \tag{6.6e}$$
$$\varphi = \frac{2(\alpha+1)}{4\alpha+5}\partial^\mu\psi_\mu + \frac{1}{2(4\alpha+5)}\partial^\mu\varphi_\mu, \tag{6.6f}$$

i.e. a one-parameter family of systems yielding a unique spin-2, unit mass field theory.

To convert this theory to the form given by Deser *et al* we must absorb the tensors $\varphi_{\mu\nu\rho}$, $G_{\mu,\nu\rho}$, φ_μ, ψ_μ into the single tensor $\Gamma_{\mu\nu,\rho}$ and $\varphi_{\mu\nu}$, φ into $h_{\mu\nu}$. $\varphi_{\mu\nu\rho}$ is identified with the symmetric traceless part of $\Gamma_{\mu\nu,\rho}$; $G_{\mu,\nu\rho}$ can be expressed in terms of $\Gamma_{\mu\nu,\rho}$ by the general result (3.3); φ_μ must be identified with $\Gamma^\rho{}_{\rho,\mu}$ and ψ_μ with $\Gamma^\rho{}_{\mu,\rho}$. $\varphi_{\mu\nu}$ and φ are identified respectively with the traceless part and trace of $h_{\mu\nu}$. With these identifications it is just a matter of algebra to convert (6.6) to the form given by Hagen (1972) for the theory of Deser *et al.* However, we shall obtain a one-parameter family of such theories, of which Hagen's form is just a special case.

The coupling of this equation to an external electromagnetic field has been considered by Federbush (1961) and Fierz and Pauli (1939), and it is known that to maintain the correct number of constraints in the theory one must add a non-minimal term to the Lagrangian (Hagen 1972). This led Hagen to conjecture that only the theories with the minimum number of components (that of § 3 in the spin-2 case) allow consistent *minimal* coupling (that is, without loss of constraints). Be that as it may, little comfort may be drawn from it, since the theory of § 3 is in any case acausal.

7. Minimal polynomial of L_0

As is well known, the L_0 matrix in (1.1) obeys a minimal polynomial of the form (Harish-Chandra 1947)

$$L_0^q(L_0^2 - 1) = 0 \tag{7.1}$$

for theories with unique mass and spin. The nilpotency index q gives a measure of the complexity of the constraints in the theory, and also the minimal polynomial of L_0 determines that of $L_\mu \partial^\mu$, which is needed in the calculation of the Klein–Gordon divisor for (1.1) (Takahashi 1969). The theories of this paper provide examples of a simple graphical technique of placing bounds on q (Cox 1981). Consider the theory of § 3. In the notation of Cox (1981) we have, from the graph of the 1 block, $d_u = 2$ and $\beta_1 = 1$, and so for this theory the general result

$$d_u + 1 \leqslant q \leqslant 2\beta_1 + 1 \tag{7.2}$$

gives by direct inspection $q = 3$.

For the theory of § 4, the 1 block gives $d_u = 3$, $\beta_1 = 2$, so for this theory we have $4 \leqslant q \leqslant 5$. Note that in this case we have in fact $2j_p - 1 < q \leqslant 2j_m - 1$ where j_p is the maximum spin of the physical state in the theory (i.e. spin 2), and j_m is the maximum spin occurring in the Lorentz representation of the theory (Cox 1981). This theory therefore exhibits Glasses pathology (Glass 1970).

For the theory of § 5 the 1 block again gives $d_u = 2$, $\beta_1 = 3$, so we obtain the bounds

$$3 \leqslant q \leqslant 7.$$

The upper bound here is hardly surprising, since the 1 block is 7×7.

The theory of § 6 is particularly interesting. Here it is the 0 block which gives the best bounds on q. From this we have $d_u = 4$, $\beta_1 = 2$ and so (7.2) gives $q = 5$. Thus, it appears that the 'standard' first-order massive spin-2 theory, constructed by analogy with the Palatini gravitational formalism, suffers from the Glass pathology, i.e. $q > 2s - 1$. It does however still satisfy $q \leqslant 2j_m - 1$.

It is interesting to note that the theory of § 3 is the unique theory for which the graphical analysis gives q = minimum = 3. This may provide a quantitative formulation for the conjecture of Hagen mentioned in § 6. Thus, we might conjecture that the least troublesome theories are those for which $q = 2s - 1$ is the only possibility for q, and that for such theories minimal coupling does not result in a loss of constraints. In this connection it is interesting that the Singh–Hagen equations for general spin s (Singh and Hagen 1974) do indeed satisfy $q = 2s - 1$ (Cox 1978, 1981), although of course their interaction consistency to loss of constraints has not yet been established.

8. Conclusions

We have shown how to obtain general first-order spin-2 theories by first identifying the possibilities by visual inspection of the graphs of the theories, and then using the tensor form to analyse the constraint structure of the various systems of equations corresponding to the possible linkage schemes. Without using RIR there are only three distinct types of first-order spin-2 theories, distinguished by their graphs, using at most third-rank tensors. One of these is equivalent to the already familiar formalism of Schwinger and Chang while the other two are new and very complicated. This complication is reflected in the possible minimal polynomials of L_0 and also in the increased complexity of the minimally coupled theory. It certainly seems unlikely that the more complicated theories improve the consistency problem of the interacting spin-2 theory.

The treatment of theories with RIR has been illustrated with an example which is equivalent to a 'standard' theory, already well known.

The simplest theory of all is the 30-component theory of Schwinger and Chang. It has the least number of components and has the best behaviour under interactions.

The possible minimal polynomials of L_0 have been investigated for the different theories. The Schwinger–Chang theory is the only theory which gives $q = 2s - 1 = 3$ unambiguously.

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